Algebraic Number Theory

Exercise Sheet 4

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Exercise 1. Let $P(X) = X^n + aX + b \in \mathbb{Z}[X]$ be an irreducible polynomial. Let $\alpha \in \mathbb{C}$ be a root of $P(X)$ and let $K = \mathbb{Q}(\alpha)$. Show that

$$D_{\mathbb{Q}}^{K}(1, \alpha, ..., \alpha^{n-1}) = (-1)^{\frac{1}{2}n(n-1)}(n^n b^{n-1} + a^n(1-n)^{n-1}).$$

Hint: Use Exercise 3.1, Sheet 3.

Exercise 2. For positive integer n let ζ_n denote a primitive n-th root of unity. Let p be an odd prime number, let $\mathbb{Q}(\zeta_p)$ be the corresponding cyclotomic field. Recall that $\mathbb{Q}(\zeta_p)/\mathbb{Q}$ is Galois field extension and $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}) = (\mathbb{Z}/p\mathbb{Z})^*$.
(1) Deduce that there is a unique intermediate field K of $\mathbb{Q}(\zeta_p)/\mathbb{Q}$, which is of degree 2 over $\mathbb{Q}$.
(2) Using Theorem 13 from the lecture and the fact, that $d_{\mathbb{Q}(\zeta_p)} = (-1)^{\frac{p-1}{2}} p^{p-2}$ (Exercise 3.2, Sheet 3), find explicitly the unique quadratic extension $K/\mathbb{Q}$ contained in $\mathbb{Q}(\zeta_p)$.
(3) Deduce that any quadratic extension of $\mathbb{Q}$ is contained inside some cyclotomic field.
Hint: Show that $\sqrt{2} \in \mathbb{Q}(\zeta_8)$ and use that $\mathbb{Q}(\zeta_n) \subset \mathbb{Q}(\zeta_{nm})$ for any positive integers n and m.

Exercise 3. Let A be a noetherian integral domain. For any non-zero element $f \in A$ denote by A_f the localization $S^{-1}A$, where $S = \{f^k \mid k \geq 0\}$.
Let $f_1, ..., f_n$ be elements from A, such that the ideal generated by $f_1, ..., f_n$ is A. Show that A is a Dedekind ring if and only if A_{f_i} is a Dedekind ring for any $i = 1, ..., n$.

Exercise 4. Let $L = \mathbb{Q}(\alpha)$ be a quadratic field extension of $\mathbb{Q}$, where $\alpha^2 = -5$. Let $\mathcal{O}_L$ be the ring of integers of L. Recall that $\mathcal{O}_L = \mathbb{Z}[\alpha]$.
(1) Show that the ideal $\rho = (2, 1+\alpha)$ is prime and not principle in $\mathcal{O}_L$.
(2) In the lecture we will see that the localization $(\mathcal{O}_L)_\rho$ is a PID. Find a generator of the ideal $\rho(\mathcal{O}_L)_\rho$.